EXTENDED SOLUTIONS OF THE HARMONIC MAP EQUATION IN THE SPECIAL UNITARY GROUP

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Abstract. We classify all harmonic maps of finite uniton number from a Riemann surface into $\mathrm{SU}(n)$ in terms of certain pieces of the Bruhat decomposition of $\Omega_{\mathrm{alg}}\mathrm{SU}(n)$, the subgroup of algebraic loops in $\mathrm{SU}(n)$. We give a description of the "Frenet frame data" for such harmonic maps in a given class.

1. Introduction

Harmonic maps from a Riemann surface into a Lie group G, with Lie algebra $\mathfrak{g}$, correspond to certain holomorphic maps, the *extended solutions*, into the group $\Omega\mathrm{G}$ of based smooth loops in G [14]. If the Fourier series associated to an extended solution Φ has finitely many terms, we say that Φ and the corresponding harmonic map have *finite uniton number*. It is well known that all harmonic maps from the two-sphere have finite uniton number [14].

When G has trivial center, Burstall and Guest [1] have classified harmonic maps with finite uniton number from a Riemann surface M into G in terms of the pieces of the Bruhat decomposition of

$$\Omega_{\mathrm{alg}}\mathrm{G} = \{\gamma \in \Omega\mathrm{G} \,|\, \gamma \text{ and } \gamma^{-1} \text{ have finite Fourier series}\}.$$

More precisely, each piece of the Bruhat decomposition coincides with the unstable manifold associated to the flow of the gradient vector field of a certain Morse-Bott function defined on the Kähler manifold $\Omega_{\mathrm{alg}}\mathrm{G}$; these unstable manifolds are parameterized by the elements of a certain integer lattice $\mathfrak{I}(\mathrm{G})$ in $\mathfrak{g}$; any extended solution with finite uniton number takes values, off a discrete subset of M, in one of these unstable manifolds, and so corresponds to some element $\xi \in \mathfrak{I}(\mathrm{G})$; when G has trivial center and maximal torus with dimension n, there is a finite subset $\Xi_{\mathrm{can}}(G)$ of the integer lattice $\mathfrak{I}(\mathrm{G})$ with 2^n elements so that any harmonic map from M to G corresponds to an extended solution with values, off a discrete subset, on the unstable manifold associated to some *canonical element* $\xi \in \Xi_{\mathrm{can}}(G)$. Among such extended solutions, a distinguished type is that of S^1-*invariant* extended solutions, which correspond to harmonic maps admitting *super-horizontal* holomorphic lifts into a certain *twistor space*. For example, all harmonic spheres in S^n and $\mathbb{C}\mathrm{P}^n$ arise in this way (see [7] and references therein).

In the present paper, we classify all harmonic maps with finite uniton number from M into the special unitary group $\mathrm{SU}(n)$ and corresponding inner symmetric spaces, the Grassmannians

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$\mathrm{Gr}(k,n)$ of k-dimensional subspaces of $\mathbb{C}^n$, in terms of certain pieces of the Bruhat decomposition of $\Omega_{\mathrm{alg}}\mathrm{SU}(n)$. For that we use the results of [6] in order to generalize the notion of canonical element of $\mathfrak{I}(\mathrm{G})$ to the case where G has not necessarily trivial center (recall that the center of $\mathrm{SU}(n)$ is isomorphic to the cyclic group $\mathbb{Z}_n$). Moreover, in the setting of the Grassmannian model for loops groups [11] we give a description of the "Frenet frame data" for such harmonic maps in a given class. The Grassmannian model for loop groups was exploited for the first time in the study of harmonic maps into the unitary group $\mathrm{U}(n)$ by Segal [12]. More recently, the Grassmannian model has been used in the study of harmonic maps into other Lie groups and their inner symmetric spaces [6, 10, 13]. We remark that Ferreira, Simões and Wood [8] established an algebraic formula for all harmonic maps with finite uniton number from a Riemann surface M into the unitary group $\mathrm{U}(n)$ in terms of freely chosen meromorphic functions on M and their derivatives. Since any such harmonic map has constant determinant, this formula can be easily applied in order to obtain all harmonic maps with finite uniton number from M into $\mathrm{SU}(n)$. However, it does not clarifies how to choose the meromorphic functions in order to produce harmonic maps associated to extended solutions in the class of a given element $\xi \in \Xi_{\mathrm{can}}(\mathrm{SU}(n))$. In this paper we shall see how to do that in the case of harmonic maps associated to S^1-invariant extended solutions.

2. Grassmannian model for loop groups

Let us start by recalling from Pressley and Segal [11] some standard definitions and facts concerning the Grassmannian model for loop groups.

Fix on $\mathbb{C}^n$ the standard complex inner product $\langle\cdot,\cdot\rangle$ and let $e_1,\dots,e_n$ be the standard basis vectors for $\mathbb{C}^n$. Given a complex subspace $V\subset\mathbb{C}^n$, we denote by π_V the orthogonal projection onto V. Let H^n be the Hilbert space of square-summable $\mathbb{C}^n$-valued functions on the circle and $\langle\cdot,\cdot\rangle_H$ the induced complex inner product. This is the closed space generated by the functions $\lambda\mapsto\lambda^i e_j$, with $i\in\mathbb{Z}$ and $j=1,\dots,n$. Consider the closed subspace H^n_+ of H^n defined by $H^n_+=\mathrm{Span}\{\lambda^i e_j\,|\,i\geq 0,\, j=1,\dots,n\}$. Let *Grass*$(H^n)$ denote the set of all closed vector subspaces $W\subset H^n$ such that: the projection map $W\to H^n_+$ is Fredholm, and the projection map $W\to H^{n\perp}_+$ is Hilbert-Schmidt; the images of the projection maps $W^\perp\to H^n_+$, $W\to H^{n\perp}_+$ are contained in $C^\infty(\mathrm{S}^1;\mathbb{C}^n)$. Define

$$\mathrm{Gr}^n=\{W\in \mathit{Grass}(H^n)\,|\,\lambda W\subseteq W\}.$$

The action of the infinite-dimensional Lie group $\Lambda\mathrm{U}(n)=\big\{\gamma: S^1\to\mathrm{U}(n)\,|\,\gamma \text{ is smooth}\big\}$ on Gr^n defined by $\gamma W=\{\gamma f\,|\,f\in W\}$ is transitive. By considering Fourier series, it is easy to see that the isotropy subgroup at H^n_+ is precisely $\mathrm{U}(n)$. Hence $\mathrm{Gr}^n\cong\Lambda\mathrm{U}(n)/\mathrm{U}(n)\cong\Omega\mathrm{U}(n)$. This homogeneous space carries a natural invariant structure of Kähler manifold.

Remark 2.1. Given $W\in\mathrm{Gr}^n$, then $\dim W\ominus\lambda W=n$, where $W\ominus\lambda W$ denotes the orthogonal complement of λW in W, and the evaluation map $\mathrm{ev}_\lambda:\mathrm{W}\ominus\lambda\mathrm{W}:\to\mathbb{C}^{\mathrm{n}}$ at $\lambda\in\mathrm{S}^1$ is a unitary isomorphism. If we choose an orthonormal basis for $W\ominus\lambda W$, $\{w_1,\dots,w_n\}$, we can put the vector-valued functions w_i side by side to form an $(n\times n)$-matrix valued function γ on S^1, that is, a loop $\gamma\in\Lambda\mathrm{U}(n)$. It can be shown that $W=\gamma H^n_+$.

A loop $\gamma\in\Omega\mathrm{U}(n)$ is said to be *algebraic* if both γ and γ^{-1} have finite Fourier series. Denote by $\Omega_{\mathrm{alg}}\mathrm{U}(n)$ the subgroup of algebraic loops. This subgroup acts on

$$\mathrm{Gr}^n_{\mathrm{alg}}=\{W\in\mathrm{Gr}^n\,|\,\lambda^k H^n_+\subseteq W\subseteq\lambda^{-k}H^n_+ \text{ for some } k\in\mathbb{N}\},$$

and we have $\mathrm{Gr}^n_{\mathrm{alg}} \cong \Omega_{\mathrm{alg}}\mathrm{U}(n)$. Given $r \leq k$, we set

$$\Omega^k_r\mathrm{U}(n) = \big\{\gamma \in \Omega_{\mathrm{alg}}\mathrm{U}(n) \,|\, \gamma(\lambda) = \sum_{i=r}^{k} \gamma_i\lambda^i\big\},$$

where the coefficients γ_i are constant $(n \times n)$-complex matrices. If G is a subgroup of $\mathrm{U}(n)$, we shall denote by $\mathrm{Gr}(\mathrm{G})$ the subspace of Gr^n that corresponds to $\Omega\mathrm{G}$ and by $\mathrm{Gr}_{\mathrm{alg}}(\mathrm{G})$ the subspace of $\mathrm{Gr}(\mathrm{G})$ that corresponds to $\Omega_{\mathrm{alg}}\mathrm{G}$.

3. The Bruhat Decomposition of $\mathrm{Gr}_{\mathrm{alg}}(\mathrm{G})$

Next we describe the Bruhat decomposition for algebraic loop groups. For more details we refer the reader to [1] and [11].

Consider a compact matrix semi-simple Lie group G. Fix a maximal torus T of G with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$, let $\Delta \subset \sqrt{-1}\mathfrak{t}^*$ be the corresponding set of roots and denote by $\mathfrak{g}_\alpha$ the root space of $\alpha \in \Delta$. The integer lattice $\mathfrak{I}(\mathrm{G}) = (2\pi)^{-1}\exp^{-1}(e) \cap \mathfrak{t}$ may be identified with the group of homomorphisms $\mathrm{S}^1 \to \mathrm{T}$, by associating to $\xi \in \mathfrak{I}(\mathrm{G})$ the homomorphism γ_ξ defined by $\gamma_\xi(\lambda) = \exp(-\sqrt{-1}\ln(\lambda)\xi)$. Let $H_1, \ldots, H_k \in \mathfrak{t}$ be dual to the positive simple roots $\alpha_1, \ldots, \alpha_k \in \Delta^+$ of $\mathfrak{g}^{\mathbb{C}}$: $\alpha_i(H_j) = \sqrt{-1}\,\delta_{ij}$. By applying the well-known formula $\mathrm{Ad}(\exp(\eta)) = \exp(\mathrm{ad}(\eta))$, for all $\eta \in \mathfrak{g}^{\mathbb{C}}$, we can easily check that the integer lattice $\mathfrak{I}(G)$ is contained in $\mathbb{Z}H_1 \oplus \ldots \oplus \mathbb{Z}H_k$. Denote by $\mathfrak{g}^\xi_i$ the $\sqrt{-1}\,i$-eigenspace of $\mathrm{ad}\xi$, with $i \in \mathbb{Z}$. We have on $\mathfrak{g}^{\mathbb{C}}$ the structure of graded Lie algebra:

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{i\in\{-r(\xi)\ldots,r(\xi)\}} \mathfrak{g}^\xi_i, \quad [\mathfrak{g}^\xi_i, \mathfrak{g}^\xi_j] \subset \mathfrak{g}^\xi_{i+j},$$

where $r(\xi) = \max\{i \,|\, \mathfrak{g}^\xi_i \neq 0\}$, and

$$\mathfrak{g}^\xi_i = \bigoplus_{\alpha(\xi)=\sqrt{-1}\,i} \mathfrak{g}_\alpha. \tag{1}$$

Set $\Lambda^+\mathrm{G}^{\mathbb{C}} = \{\gamma : S^1 \to \mathrm{G}^{\mathbb{C}} \,|\, \gamma$ extends holomorphically for $|\lambda| \leq 1\}$. For each $\xi \in \mathfrak{I}(\mathrm{G})$, we write $\Omega_\xi = \{g\gamma_\xi g^{-1} \,|\, g \in \mathrm{G}\}$, the conjugacy class of homomorphisms $\mathrm{S}^1 \to \mathrm{G}$ which contains γ_ξ. This is a complex homogeneous space:

$$\Omega_\xi \cong \mathrm{G}^{\mathbb{C}}/\mathrm{P}_\xi, \text{ with } \mathrm{P}_\xi = \mathrm{G}^{\mathbb{C}} \cap \gamma_\xi\Lambda^+\mathrm{G}^{\mathbb{C}}\gamma_\xi^{-1}.$$

Taking account that $\gamma_\xi X_j \gamma_\xi^{-1} = \lambda^j X_j$ for each $X_j \in \mathfrak{g}^\xi_j$ (this is a direct consequence of the formula $\mathrm{Ad}(\exp(\eta)) = \exp(\mathrm{ad}(\eta))$, for all $\eta \in \mathfrak{g}^{\mathbb{C}}$), one can easily check that the Lie algebra of the isotropy subgroup P_ξ is precisely the parabolic subalgebra $\mathfrak{p}_\xi = \bigoplus_{i\leq 0}\mathfrak{g}^\xi_i$ induced by ξ.

Now, fix a positive set of roots $\Delta^+ \subset \Delta$ and set $\mathfrak{I}'(\mathrm{G}) = \{\xi \in \mathfrak{I}(\mathrm{G}) | \, \alpha(\xi) \geq 0$ for all $\alpha \in \Delta^+\}$. We have:

Theorem 3.1. [11] *Bruhat decomposition:* $\mathrm{Gr}_{\mathrm{alg}}(\mathrm{G}) = \bigcup_{\xi\in\mathfrak{I}'(\mathrm{G})} \Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}}\gamma_\xi H^n_+$.

Define $U_\xi(\mathrm{G}) \subset \Omega_{\mathrm{alg}}\mathrm{G}$ by $U_\xi(\mathrm{G})H^n_+ = \Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}}\gamma_\xi H^n_+$. This is also a complex homogeneous space of $\Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}}$ with isotropy subgroup at γ_ξ given by $\Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}} \cap \gamma_\xi\Lambda^+\mathrm{G}^{\mathbb{C}}\gamma_\xi^{-1}$. Moreover, $U_\xi(\mathrm{G})$ carries the structure of holomorphic vector bundle over Ω_ξ whose bundle map $u_\xi : U_\xi(\mathrm{G}) \to \Omega_\xi$

is precisely the natural map $[\gamma] \mapsto [\gamma(0)]$. The holomorphic tangent bundle of $U_\xi(\mathrm{G})$ is given by

$$T^{1,0}U_\xi(\mathrm{G}) \cong \Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}} \times_{\Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}} \cap \gamma_\xi \Lambda^+\mathrm{G}^{\mathbb{C}}\gamma_\xi^{-1}} \Lambda^+_{\mathrm{alg}}\mathfrak{g}^{\mathbb{C}} / \Lambda^+_{\mathrm{alg}}\mathfrak{g}^{\mathbb{C}} \cap \gamma_\xi \Lambda^+ \mathfrak{g}^{\mathbb{C}} \gamma_\xi^{-1} \tag{2}$$

In terms of the Grassmannian model, the bundle map $u_\xi : U_\xi(\mathrm{G}) \to \Omega_\xi$ can be described as follows. Take $\gamma \in U_\xi(\mathrm{G})$ and $W = \gamma H^n_+ \in \mathrm{Gr}_{\mathrm{alg}}(\mathrm{G})$, with $\lambda^r H^n_+ \subset W \subset \lambda^{-s} H^n_+$. Fix $\Psi \in \Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}}$ such that $W = \Psi\gamma_\xi H^n_+$. Write

$$\gamma_\xi H^n_+ = \lambda^{-s} A^\xi_{-s} + \ldots + \lambda^{r-1} A^\xi_{r-1} + \lambda^r H^n_+,$$

where the subspaces A^ξ_i define a flag

$$\{0\} = A^\xi_{-s-1} \subsetneq A^\xi_{-s} \subseteq A^\xi_{-s+1} \subseteq \ldots \subseteq A^\xi_{r-1} \subsetneq A^\xi_r = \mathbb{C}^n. \tag{3}$$

Set $A_i = \Psi(0)A^\xi_i = p_i(W \cap \lambda^i H^n_+)$, where $p_i : H^n \to \mathbb{C}^n$ is defined by

$$p_i\left(\sum \lambda^j a_j\right) = a_i. \tag{4}$$

Then

$$u_\xi(W) = \lambda^{-s} A_{-s} + \ldots + \lambda^{r-1} A_{r-1} + \lambda^r H^n_+. \tag{5}$$

Following [6], consider the partial order $\preceq$ over $\mathfrak{I}'(\mathrm{G})$ defined by: $\xi \preceq \xi'$ if $\mathfrak{p}^\xi_i \subseteq \mathfrak{p}^{\xi'}_i$ for all $i \geq 0$, where $\mathfrak{p}^\xi_i = \bigoplus_{j \leq i} \mathfrak{g}^\xi_j$. Given two elements $\xi, \xi' \in \mathfrak{I}'(\mathrm{G})$ such that $\xi \preceq \xi'$, it can be shown [6] that

$$\Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}} \cap \gamma_\xi \Lambda^+ \mathrm{G}^{\mathbb{C}} \gamma_\xi^{-1} \subseteq \Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}} \cap \gamma_{\xi'} \Lambda^+ \mathrm{G}^{\mathbb{C}} \gamma_{\xi'}^{-1}.$$

This allows us to define, for $\xi \preceq \xi'$, a $\Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}}$-invariant fibre bundle morphism $\mathcal{U}_{\xi,\xi'} : U_\xi \to U_{\xi'}$ by

$$\mathcal{U}_{\xi,\xi'}(\Psi\gamma_\xi H^n_+) = \Psi\gamma_{\xi'} H^n_+, \quad \Psi \in \Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}}.$$

Since the holomorphic structures on $U_\xi(\mathrm{G})$ and $U_{\xi'}(\mathrm{G})$ are induced by the holomorphic structure on $\Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}}$, the fibre-bundle morphism $\mathcal{U}_{\xi,\xi'}$ is holomorphic.

4. Harmonic maps into a Lie group

4.1. **Extended solutions.** Let M be a Riemann surface and $\varphi : M \to \mathrm{G} \subseteq \mathrm{U}(n)$ a map into a compact matrix Lie group. Equip G with a bi-invariant metric. Define $\alpha = \varphi^{-1} d\varphi$ and let $\alpha = \alpha' + \alpha''$ be the type decomposition of α into $(1,0)$ and $(0,1)$-forms. It is well known [14] that $\varphi : M \to \mathrm{G}$ is harmonic if and only if the loop of 1-forms given by

$$\alpha_\lambda = \frac{1 - \lambda^{-1}}{2}\alpha' + \frac{1 - \lambda}{2}\alpha'' \tag{6}$$

satisfies the Maurer-Cartan equation $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ for each $\lambda \in \mathrm{S}^1$. Then, if M is simply connected and φ is harmonic, we can integrate to obtain a map $\Phi : M \to \Omega\mathrm{G}$, the *extended solution* associated to φ, such that $\alpha_\lambda = \Phi_\lambda^{-1} d\Phi_\lambda$ and $\Phi_{-1} = \varphi$. Conversely, if $\Phi : M \to \Omega\mathrm{G}$ is an extended solution, that is if it integrates a loop of 1-form of the form (6), then $\varphi = \Phi_{-1} : M \to \mathrm{G}$ is harmonic.

Theorem 4.1. [1] *Let $\Phi : M \to \Omega_{\mathrm{alg}}\mathrm{G}$ be an extended solution. Then there exists some $\xi \in \mathfrak{I}'(\mathrm{G})$, and some discrete subset D of M, such that $\Phi(M \setminus D) \subseteq U_\xi(\mathrm{G})$.*

Given a smooth map $\Phi : M \setminus D \to U_\xi(\mathrm{G})$, consider $\Psi : M \setminus D \to \Lambda^+_{\mathrm{alg}}\mathrm{G}^{\mathbb{C}}$ such that $\Phi H^n_+ = \Psi\gamma_\xi H^n_+$. Clearly, $\Psi\gamma_\xi = \Phi b$ for some $b : M \setminus D \to \Lambda^+_{\mathrm{alg}}G^{\mathbb{C}}$. Write

$$\Psi^{-1}\Psi_z = \sum_{i\geq 0} X'_i\lambda^i, \quad \Psi^{-1}\Psi_{\bar z} = \sum_{i\geq 0} X''_i\lambda^i.$$

Proposition 4.4 in [1] establishes that Φ is an extended solution if, and only if,

$$\mathrm{Im}X'_i \subset \mathfrak{p}^\xi_{i+1}, \quad \mathrm{Im}X''_i \subset \mathfrak{p}^\xi_i, \tag{7}$$

where $\mathfrak{p}^\xi_i = \bigoplus_{j\leq i}\mathfrak{g}^\xi_j$. The second condition says that $\Phi : M \setminus D \to U_\xi(\mathrm{G})$ is holomorphic.

The bundle morphism $\mathcal{U}_{\xi,\xi'}$ and the bundle map u_ξ are well behaved with respect to extended solutions:

Theorem 4.2. [6] *Given an extended solution $\Phi : M \setminus D \to U_\xi(\mathrm{G})$ and an element $\xi' \in \mathfrak{I}'(\mathrm{G})$ such that $\xi \preceq \xi'$, then $\mathcal{U}_{\xi,\xi'}(\Phi) = \mathcal{U}_{\xi,\xi'} \circ \Phi : M \setminus D \to U_{\xi'}(\mathrm{G})$ is a new extended solution.*

Theorem 4.3. [1] *If $\Phi : M \setminus D \to U_\xi(\mathrm{G})$ is an extended solution, then $u_\xi \circ \Phi : M \setminus D \to \Omega_\xi$ is an extended solution.*

An S^1-*invariant extended solution* is an extended solution which takes values in Ω_ξ, for some $\xi \in \mathfrak{I}'(\mathrm{G})$.

4.2. **Harmonic maps into inner G-symmetric spaces.** Given a compact (connected) Lie group G, each connected component of $\sqrt{e} = \{g \in \mathrm{G} \mid g^2 = e\}$ is a compact inner symmetric space [1]. Conversely, any compact (connected) inner G-symmetric space may be immersed in G as a connected component of $\sqrt{e}$. Moreover, the embedding of each component of $\sqrt{e}$ in G is totally geodesic. Hence harmonic maps into G-inner symmetric spaces can be viewed as special harmonic maps into G.

As in [1], define the involution $\mathcal{I} : \Omega\mathrm{G} \to \Omega\mathrm{G}$ by $\mathcal{I}(\gamma)(\lambda) = \gamma(-\lambda)\gamma(-1)^{-1}$. Write

$$\Omega^{\mathcal{I}}\mathrm{G} = \{\gamma \in \Omega\mathrm{G} \mid \mathcal{I}(\gamma) = \gamma\}$$

for the fixed set of $\mathcal{I}$. Let M be a Riemann surface and $\Phi : M \to \Omega^{\mathcal{I}}\mathrm{G}$ an extended solution. Then $\varphi = \Phi_{-1}$ defines a harmonic map from M into a connected component of $\sqrt{e}$. Conversely, if $\varphi : M \to \sqrt{e}$ is a harmonic map, there exists an extended solution $\Phi : M \to \Omega^{\mathcal{I}}\mathrm{G}$ such that $\varphi = \Phi_{-1}$. Under the identification $\Omega\mathrm{G} \cong \mathrm{Gr}(\mathrm{G})$, $\mathcal{I}$ induces an involution on $\mathrm{Gr}(\mathrm{G})$, that we shall also denote by $\mathcal{I}$, and $\Omega^{\mathcal{I}}\mathrm{G}$ can be identified with

$$\mathrm{Gr}^{\mathcal{I}}(\mathrm{G}) = \{W \in \mathrm{Gr}(\mathrm{G}) \mid \text{ if } s(\lambda) \in W \text{ then } s(-\lambda) \in W\}.$$

Corresponding to the extended solution $\Phi : M \to \Omega^{\mathcal{I}}\mathrm{G}$, consider $W = \Phi H_+ : M \to \mathrm{Gr}^{\mathcal{I}}(\mathrm{G})$.

For each $\xi \in \mathfrak{I}'(\mathrm{G})$ we can associate the symmetric space $N_\xi = \{g\gamma_\xi(-1)g^{-1} \mid g \in G\}$. If an extended solution takes values in $U^{\mathcal{I}}_\xi(\mathrm{G}) = U_\xi(\mathrm{G}) \cap \Omega^{\mathcal{I}}\mathrm{G}$, then the corresponding harmonic map takes values in N_ξ. Observe that, for ξ and ξ' in $\mathfrak{I}'(\mathrm{G})$, if $\xi - \xi' \in \mathfrak{I}^2(\mathrm{G}) := \pi^{-1}\exp^{-1}(e) \cap \mathfrak{t}$, then $N_\xi = N_{\xi'}$. Moreover, as shown in [6], if $\xi \preceq \xi'$, then $\mathcal{U}_{\xi,\xi'}(U^{\mathcal{I}}_\xi(\mathrm{G})) \subset U^{\mathcal{I}}_{\xi'}(\mathrm{G})$. To sum up, if we define a new partial order $\preceq_{\mathcal{I}}$ in $\mathfrak{I}'(\mathrm{G})$ by

$$\xi \preceq_{\mathcal{I}} \xi' \quad \text{if } \xi \preceq \xi' \text{ and } \xi - \xi' \in \mathfrak{I}^2(\mathrm{G}),$$

the following holds:

Proposition 4.4. *If $\xi \preceq_{\mathcal{I}} \xi'$, then $\mathcal{U}_{\xi,\xi'}(U^{\mathcal{I}}_\xi(\mathrm{G})) \subset U^{\mathcal{I}}_{\xi'}(\mathrm{G})$ and $N_\xi = N_{\xi'}$.*

4.3. **Extended solutions from the Grassmannian point of view.** Let $W : M \to \mathrm{Gr}(\mathrm{G})$ correspond to a smooth map $\Phi : M \to \Omega\mathrm{G}$ under the identification $\Omega\mathrm{G} \cong \mathrm{Gr}(\mathrm{G})$, that is $W = \Phi H^n_+$. Segal [12] has observed that Φ is an extended solution if, and only if, W is a solution of equations:

$$W_z \subseteq \lambda^{-1} W, \tag{8}$$

$$W_{\bar{z}} \subseteq W. \tag{9}$$

Condition (8) means that, in any local complex coordinate z, $\frac{\partial s}{\partial z}(z)$ is contained in the subspace $\lambda^{-1}W(z)$ of H^n, for every (smooth) map $s : M \to H^n$ such that $s(z) \in W(z)$. Inspired by [2] (Section F of Chapter 8), we call (8) the *pseudo-horizontality* condition. Condition (9) is interpreted in a similar way and states that W is a holomorphic vector subbundle of $M \times H^n$.

Remark 4.5. Consider some discrete set $D \subset M$, an element $\xi \in \mathfrak{I}'(\mathrm{G})$ and an extended solution $\Phi : M \setminus D \to U_\xi(\mathrm{G})$. As explained in Remark 2.5 of [5], the bundle $W = \Phi H^n_+$ can be extended holomorphically to M, and, consequently, Φ defines a global extended solution from M to $\Omega_{\mathrm{alg}}\mathrm{G}$.

If $\Phi : M \setminus D \to U_\xi(\mathrm{G})$ is an extended solution and $W = \Phi H^n_+$, then $u_\xi(W) = u_\xi \circ \Phi H^n_+$ is given pointwise by (5) and we get holomorphic subbundles A_i of the trivial bundle $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$ such that

$$0 \subsetneq A_{-s} \subseteq \ldots \subseteq A_{r-1} \subsetneq A_r = \underline{\mathbb{C}}^n. \tag{10}$$

The pseudo-horizontally condition implies that $A_{iz} \subseteq A_{i+1}$, that is, following again the terminology of [2], the flag of holomorphic vector bundles (10) is *super-horizontal.*

4.4. **Normalization of extended solutions.** The following theorem, which is a generalization of Theorem 4.5 in [1], is fundamental to the classification of extended solutions.

Theorem 4.6. [6] *Let $\Phi : M \setminus D \to U_\xi(\mathrm{G})$ be an extended solution. Take $\xi' \in \mathfrak{I}'(\mathrm{G})$ such that $\xi \preceq \xi'$ and $\mathfrak{g}_0^\xi = \mathfrak{g}_0^{\xi'}$. Then there exists some constant loop $\gamma \in \Omega_{\mathrm{alg}}\mathrm{G}$ such that $\gamma\Phi : M \setminus D \to U_{\xi'}(\mathrm{G})$.*

A similar statement holds for extended solutions associated to harmonic maps into symmetric spaces:

Theorem 4.7. *Let $\Phi : M \setminus D \to U^{\mathcal{I}}_\xi(\mathrm{G})$ be an extended solution. Take $\xi' \in \mathfrak{I}'(\mathrm{G})$ such that $\xi \preceq_{\mathcal{I}} \xi'$. Then there exists some constant loop $\gamma \in \Omega^{\mathcal{I}}_{\mathrm{alg}}\mathrm{G}$ such that $\gamma\Phi : M \setminus D \to U^{\mathcal{I}}_{\xi'}(\mathrm{G})$.*

Proof. We can write $\Phi H^n_+ = \Psi\gamma_\xi H^n_+$, where $\Psi : M \setminus D \to \Lambda^+_{\mathrm{alg}} G^{\mathbb{C}}$ contains only even powers of λ, and consequently $\Psi^{-1}\Psi_z = \sum_{i \geq 0} X'_i \lambda^i$ contains only even powers of λ. The extended solution equation (7) gives $\mathrm{Im}\, X'_{2j} \subset \mathfrak{p}^\xi_{2j+1}$ for all $j \geq 0$. Set $\hat{\xi} = \xi - \xi' \in \mathfrak{I}^2(\mathrm{G})$. Clearly $\xi \preceq \hat{\xi}$, hence $\mathfrak{p}^\xi_{2j+1} \subseteq \mathfrak{p}^{\hat{\xi}}_{2j+1}$ for all $j \geq 0$. On the other hand, since $\alpha(\hat{\xi}) = 2\sqrt{-1}\mathbb{Z}$ for any positive root α, we have $\mathfrak{g}^{\hat{\xi}}_{2j+1} = 0$ and, consequently, $\mathfrak{p}^{\hat{\xi}}_{2j+1} = \mathfrak{p}^{\hat{\xi}}_{2j}$. Hence,

$$\mathrm{Im}\, \Psi^{-1}\Psi_z \subseteq \bigoplus_{j \geq 0} \mathfrak{p}^\xi_{2j+1}\lambda^{2j} \subseteq \bigoplus_{j \geq 0} \mathfrak{p}^{\hat{\xi}}_{2j}\lambda^{2j} \subseteq \Lambda^+_{\mathrm{alg}}\mathfrak{g}^{\mathbb{C}} \cap \gamma_{\hat{\xi}}\Lambda^+\mathfrak{g}^{\mathbb{C}}\gamma_{\hat{\xi}}^{-1}.$$

Taking account (2), we conclude that $\mathcal{U}_{\xi,\hat{\xi}}(\Phi)$ is anti-holomorphic. On the other hand, since any extended solution is holomorphic and Φ is an extended solution, Theorem 4.2 asserts that

$\mathcal{U}_{\xi,\hat{\xi}}(\Phi)$ is also holomorphic. Being both holomorphic and anti-holomorphic, it must be equal to a constant loop γ^{-1}. By Proposition 4.4 we have $\gamma^{-1} \in \Omega^{\mathcal{I}}_{\mathrm{alg}}\mathrm{G}$. Write $\Psi\gamma_{\hat{\xi}} = \gamma^{-1}b$, for some map $b : M \to \Lambda^{+}\mathrm{G}$. Then

$$\Phi H^n_+ = \Psi\gamma_\xi H^n_+ = \gamma^{-1}b\gamma_{\hat{\xi}}^{-1}\gamma_\xi H^n_+ = \gamma^{-1}b\gamma_{\xi'}H^n_+,$$

which implies that $\gamma\Phi$ takes values in $U^{\mathcal{I}}_{\xi'}(\mathrm{G})$. □

Given $\xi = \sum n_i H_i$ and $\xi' = \sum n'_i H_i$ in $\mathfrak{I}'(\mathrm{G})$, we have $n_i, n'_i \geq 0$ and observe that $\xi \preceq \xi'$ if and only if $n'_i \leq n_i$ for all i. For each $I \subseteq \{1, \ldots, k\}$, define the cone

$$\mathfrak{C}_I = \Big\{ \sum_{i=1}^{k} n_i H_i |\, n_i \geq 0,\, n_j = 0 \text{ iff } j \notin I \Big\}.$$

Definition 4.8. *Let $\xi \in \mathfrak{I}'(\mathrm{G}) \cap \mathfrak{C}_I$. We say that ξ is a* I-canonical element *of $\mathfrak{g}$ with respect to Δ^+ if it is a maximal element of $(\mathfrak{I}'(\mathrm{G}) \cap \mathfrak{C}_I, \preceq)$, that is, if $\xi \preceq \xi'$ and $\xi' \in \mathfrak{I}'(\mathrm{G}) \cap \mathfrak{C}_I$ then $\xi = \xi'$. Similarly, we say that ξ is a* symmetric canonical element *of $\mathfrak{g}$ with respect to Δ^+ if it is a maximal element of $(\mathfrak{I}'(\mathrm{G}), \preceq_{\mathcal{I}})$*

When G has trivial center, which is the case considered in [1], the duals $H_1, \ldots, H_k$ belong to the integer lattice. Then, for each I there exists a unique I-canonical element, which is given by $\xi_I = \sum_{i \in I} H_i$.

Theorem 4.9. *Let $\Phi : M \to \Omega_{\mathrm{alg}}\mathrm{G}$ be an extended solution. There exist a constant loop $\gamma \in \Omega_{\mathrm{alg}}\mathrm{G}$, a subset $I \subseteq \{1, \ldots, k\}$, a I-canonical element ξ' and a discrete subset $D \subset M$, such that $\gamma\Phi(M \setminus D) \subseteq U_{\xi'}(\mathrm{G})$.*

Proof. Take $D \subset M$ and $\xi \in \mathfrak{I}'(\mathrm{G})$ in the conditions of Theorem 4.1. Write $\xi = \sum_{i=1}^{k} n_i H_i$, with $n_i \geq 0$, and set $I = \{i | n_i > 0\}$. By Zorn's lemma, there certainly exists a I-canonical element ξ' such that $\xi \preceq \xi'$. On the other hand, from (1) we see that $\mathfrak{g}_0^{\xi} = \mathfrak{g}_0^{\xi'}$. Hence the result follows from Theorem 4.6. □

Theorem 4.10. *Let $\Phi : M \to \Omega^{\mathcal{I}}_{\mathrm{alg}}\mathrm{G}$ be an extended solution with values in $U^{\mathcal{I}}_{\xi}(\mathrm{G})$, for some $\xi \in \mathfrak{I}'(\mathrm{G})$, off a discrete set D. There exist a constant loop $\gamma \in \Omega^{\mathcal{I}}_{\mathrm{alg}}\mathrm{G}$ and a symmetric canonical element ξ' such that $\gamma\Phi(M \setminus D) \subseteq U^{\mathcal{I}}_{\xi'}(\mathrm{G})$ and $N_\xi = N_{\xi'}$.*

Proof. By Zorn's lemma, there certainly exists a symmetric canonical element ξ' such that $\xi \preceq_{\mathcal{I}} \xi'$. The result follows from Proposition 4.4 and Theorem 4.7. □

4.5. **Frenet frame data for extended solutions into $\Omega_{\mathrm{alg}}\mathrm{U}(n)$.** Given a finite collection $\{s_j\}$ of meromorphic sections of the trivial bundle $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$, we obtain an holomorphic bundle away from a discrete subset of M, and we can fill in holes to extend it to subbundle E over M of $\underline{\mathbb{C}}^n$. In this case, we denote $E = \mathrm{Span}\{s_j\}$. Reciprocally, any holomorphic subbundle E of $\underline{\mathbb{C}}^n$ has a global meromorphic frame $\{s_1, \ldots, s_k\}$, with $k = \operatorname{rank} E$, as explained in [13]. For $i > 0$, the (i)*-th osculating bundle* of E is the subbundle $E^{(i)}$ of $\underline{\mathbb{C}}^n$ spanned by the local holomorphic sections of E and their derivatives up to i. We also define the $(-i)$*-th osculating bundle* of E as the subbundle $E^{(-i)}$ of $\underline{\mathbb{C}}^n$ spanned by the local holomorphic sections of E whose derivatives up to i are also local sections of E. Let $g_E = \operatorname{rank} E^{(1)} - \operatorname{rank} E$ and r_E be the remainder of the positive integer division of $\operatorname{rank} E$ by g_E: $\operatorname{rank} E = q_E g_E + r_E$.

As Guest [9] has observed, any smooth map $W : M \to \mathrm{Gr}^n$ corresponding to an extended solution $\Phi : M \to \Omega_r^k \mathrm{U}(n)$ is *generated* by a certain holomorphic subbundle X, a *Frenet frame* of Φ, of the trivial bundle $M \times \lambda^r H_+/\lambda^k H_+$ by setting

$$W = X + \lambda X^{(1)} + \ldots + \lambda^{k-r-1} X^{(k-r-1)} + \lambda^k H_+. \tag{11}$$

Hence any extended solution $\Phi : M \to \Omega_{\mathrm{alg}} \mathrm{U}(n)$ can be obtained by applying a finite number of algebraic operations on sets of meromorphic functions on M, since X can be chosen arbitrarily. In [8, 13] the authors established explicit algebraic formulae relating Frenet frames X with different classes of uniton factorizations of harmonic maps. Next we will give a description of the Frenet frames associated to extended solutions with values in a fixed piece $U_\xi(\mathrm{U}(n))$ of the Bruhat decomposition of $\Omega_{\mathrm{alg}} \mathrm{U}(n)$ and we establish a pure algebraic method to obtain all S^1-invariant extended solutions with values in a fixed Ω_ξ.

Choose a local complex coordinate z and a local section s of E. Differentiating $\pi_E^\perp(s) = 0$, where $\pi_E^\perp$ is the orthogonal projection onto $E^\perp$, we get $\pi_E^\perp(s_z) = -(\pi_E^\perp)_z(s)$. Hence the association $s \mapsto \pi_E^\perp(s_z)$ defines a local vector bundle morphism $\mathcal{A}_E : E \to E^\perp$, which, following [3], we call the *∂'-second fundamental form* of E in $\mathbb{C}^n$, whose kernel and image do not depend on the choice of the local coordinate z. It follows from the linearity of the ∂'-second fundamental form that:

Lemma 4.11. *Let E be a holomorphic vector subbundle of $\underline{\mathbb{C}}^n$.*

a) *For all $i \geq 1$, $E^{(-i)} = \ker \mathcal{A}_{E^{(-i+1)}}$ is locally spanned by those sections s of E solving the following system of algebraic linear equations: $(\pi_{E^{(-j)}}^\perp)_z(s) = 0$ for all $j = 0, \ldots, i-1$;*
b) *$g_{E^{(i)}} \leq g_E$ and $\operatorname{rank} E^{(i)} \leq \operatorname{rank} E + i g_E$ for all $i \geq 1$ (the equalities hold for $i = 1$);*
c) *$g_{E^{(-i)}} \leq g_E$ and $\operatorname{rank} E^{(-i)} \geq \operatorname{rank} E - i g_E$ for all $i \geq 1$(the equalities hold for $i = 1$);*
d) *For each $g \geq g_E$, there exists a super-horizontal flag of holomorphic subbundles*

$$E_{-q} \subsetneq E_{-q+1} \subsetneq \ldots \subsetneq E_{-1} \subsetneq E_0 = E, \tag{12}$$

such that $\operatorname{rank} E_{-i} = \operatorname{rank} E - ig$, where the integer $q \leq q_E$ is the quotient of the positive integer division of $\operatorname{rank} E$ by g: $\operatorname{rank} E = qg + r$.

Proof. For example, since $E^{(-i-1)} = \ker \mathcal{A}_{E^{(-i)}}$, we have, for all $i \geq 0$,

$$g_{E^{(-i)}} = \operatorname{rank} \operatorname{im} \mathcal{A}_{E^{(-i)}} = \operatorname{rank} \operatorname{coim} \mathcal{A}_{E^{(-i)}} = \operatorname{rank} E^{(-i)} - \operatorname{rank} E^{(-i-1)}.$$

On the other hand, since the image of $\mathcal{A}_{E^{(-i-1)}}$ is contained in $E^{(-i)} \ominus E^{(-i-1)}$, for all $i \geq 0$, we also have $g_{E^{(-i-1)}} \leq \operatorname{rank} E^{(-i)} - \operatorname{rank} E^{(-i-1)}$. Hence, for all $i \geq 0$, $g_{E^{(-i)}} \leq g_E$.

To construct a flag (12), start by taking an arbitrary holomorphic subbundle $E_{-1} \subseteq E^{(-1)}$ with $\operatorname{rank} E_{-1} = \operatorname{rank} E - g \leq \operatorname{rank} E - g_E = \operatorname{rank} E^{(-1)}$. Clearly,

$$g_{E_{-1}} = \operatorname{rank} E_{-1}^{(1)} - \operatorname{rank} E_{-1} \leq \operatorname{rank} E - \operatorname{rank} E_{-1} = g. \tag{13}$$

Hence

$$\operatorname{rank} E_{-1}^{(-1)} = \operatorname{rank} E_{-1} - g_{E_{-1}} \geq \operatorname{rank} E - 2g,$$

and we see that there exists a holomorphic subbundle E_{-2} of $E_{-1}^{(-1)}$ with $\operatorname{rank} E_{-2} = \operatorname{rank} E - 2g$. Proceeding recursively we find after q steps a super-horizontal flag of holomorphic subbundles (12). □

The following construction is fundamental for our purposes:

Lemma 4.12. *Let $T \subset E$ be two holomorphic subbundles of $\underline{\mathbb{C}}^n$. Fix a positive integer g, with $g \geq \max\{g_T, g_E\}$, and assume that, for some $i, j \geq 0$, we have $T^{(j)} \subset E^{(-i)}$. Given an integer d with $\operatorname{rank} T^{(j)} < d < \operatorname{rank} E^{(-i)}$, any holomorphic subbundle F satisfying $T^{(j)} \subset F \subset E^{(-i)}$, $\operatorname{rank} F = d$, and $g_F \leq g$, arises as follows:*

a) *Set $k_0 = \max\{k \mid d - kg > \operatorname{rank} T^{(j-k)}\}$ and $r_0 = d - k_0 g - \operatorname{rank} T^{(j-k_0)}$. Choose r_0 linearly independent meromorphic sections $s_1, \ldots, s_{r_0}$ of $E^{(-i-k_0)}$ so that the holomorphic vector bundle*

$$F_{-k_0} = T^{(j-k_0)} + \operatorname{Span}\{s_1, \ldots, s_{r_0}\} \tag{14}$$

has rank $d - k_0 g$. Independently of the choice of these meromorphic sections, we have $g_{F_{-k_0}} \leq g$.

b) *Choose $r_1 = d - (k_0 - 1)g - \operatorname{rank} F^{(1)}_{-k_0}$ meromorphic sections $s_{r_0+1}, \ldots, s_{r_0+r_1}$ of $E^{(-i-k_0+1)}$ so that the holomorphic vector subbundle*

$$F_{-k_0+1} = F^{(1)}_{-k_0} + \operatorname{Span}\{s_{r_0+1}, \ldots, s_{r_0+r_1}\}$$

has rank $d - (k_0 - 1)g$. We have $g_{F_{-k_0+1}} \leq g$.

c) *Repeat this procedure k_0 times to find a super-horizontal flag of holomorphic subbundles $F_{-k_0} \subsetneq \ldots \subsetneq F_{-1} \subsetneq F_0 = F$, with*

$$F_{-k_0+l} = F^{(1)}_{-k_0+l-1} + \operatorname{Span}\{s_{r_0+\ldots+r_{l-1}+1}, \ldots, s_{r_0+\ldots+r_{l-1}+r_l}\}, \tag{15}$$

$$r_l = d - (k_0 - l)g - \operatorname{rank} F^{(1)}_{-k_0+l-1} \text{ and } \operatorname{rank} F_{-k_0+l} = d - (k_0 - l)g.$$

Proof. Since $d < \operatorname{rank} E^{(-i)}$ and $g_{E^{(-i)}} \leq g_E \leq g$, by Lemma 4.11 inequalities

$$d - kg < \operatorname{rank} E^{(-i)} - k g_{E^{(-i)}} \leq \operatorname{rank} E^{(-i-k)}$$

hold for each $k \geq 0$. Hence we can always take $r_0 \geq 0$ linearly independent meromorphic sections of $E^{(-i-k_0)}$ so that F_{-k_0} defined by (14) has rank $d - k_0 g$. We have to check now that $g_{F_{-k_0}} \leq g$. By definition of k_0 we have $d - (k_0 + 1)g \leq \operatorname{rank} T^{(j-k_0-1)}$. Then,

$$\begin{aligned} g_{F_{-k_0}} &\leq g_{T^{(j-k_0)}} + r_0 = g_{T^{(j-k_0)}} + d - k_0 g - \operatorname{rank} T^{(j-k_0)} \\ &\leq g_{T^{(j-k_0)}} + g - (\operatorname{rank} T^{(j-k_0)} - \operatorname{rank} T^{(j-k_0-1)}) = g_{T^{(j-k_0)}} + g - g_{T^{(j-k_0)}} = g. \end{aligned}$$

Since $F_{-k_0} \subseteq \ker \mathcal{A}_{F_{-k_0+1}}$, then $g_{F_{-k_0+1}} \leq \operatorname{rank} F_{-k_0+1} - \operatorname{rank} F_{-k_0} = g$. On the other hand, it is clear that $r_1 \geq 0$. Hence the construction of item b) is possible and we can proceed recursively until find a super-horizontal flag of holomorphic subbundles $F_{-k_0} \subsetneq \ldots \subsetneq F_{-1} \subsetneq F_0 = F$, with F_{-k_0+l} given by (15), where F is certainly in the required conditions.

Reciprocally, any F as required certainly arises in this way. In fact, by Lemma 4.11 there always exists a super-horizontal flag of holomorphic subbundles $F_{-q} \subsetneq \ldots \subsetneq F_{-k_0} \subsetneq \ldots \subsetneq F_{-1} \subsetneq F_0 = F$, with $k_0 = \max\{k \mid d - kg > \operatorname{rank} T^{(j-k)}\}$. We can choose such sequence so that $T^{(j-k_0)} \subsetneq F_{-k_0}$.

□

Now we are in conditions to establish an algorithm to obtain all S^1-invariant extended solutions with values in a given Ω_ξ.

Theorem 4.13. *Fix $\xi \in \mathfrak{I}(\mathrm{U}(n))$ and consider the corresponding flag* (3)*. Set $d_i = \dim A_i^\xi$ and $h_i = d_{i+1} - d_i$. Any super-horizontal flag of holomorphic vector subbundles*

$$\{0\} = A_{-r-1} \subsetneq A_{-r} \subseteq \ldots \subseteq A_{k-1} \subsetneq A_k = \underline{\mathbb{C}}^n \tag{16}$$

with $\operatorname{rank} A_i = d_i$ *arises as follows:*

a) *Set $l = \min\{h_i \,|\, i = -r-1, \ldots, k-1\}$ and $m = \max\{i \,|\, l = h_i\}$. Apply Lemma 4.12 (with $T = \{0\}$, $E = \underline{\mathbb{C}}^n$, $d = d_m$ and $g = l$) to find A_m.*

b) *Set $l_1 = \min\{h_i \,|\, -r-1 \leq i < m\}$, $m_1 = \max\{i \,|\, l_1 = h_i, -r-1 \leq i < m\}$. Apply Lemma 4.12 (with $T = \{0\}$, $E = A_m$, $d = d_{m_1}$ and $g = l_1$) to find $A_{m_1} \subseteq A_m^{(m_1 - m)}$.*

c) *Set $l_{\hat{1}} = \min\{h_i \,|\, m < i \leq k-1\}$ and $m_{\hat{1}} = \max\{i \,|\, l_{\hat{1}} = h_i, m < i \leq k-1\}$, and apply Lemma 4.12 (with $T = A_m$, $E = \underline{\mathbb{C}}^n$, $d = d_{m_{\hat{1}}}$ and $g = l_{\hat{1}}$) to find $A_{m_{\hat{1}}} \supseteq A_m^{(m_{\hat{1}} - m)}$.*

d) *Proceed recursively until obtain a flag of the form* (16)*.*

Remark 4.14. In [1], the authors introduce a method to obtain super-horizontal flags of holomorphic subspaces associated to a given element $\xi \in \mathfrak{I}'(G)$. However, their method involves integration of meromorphic functions.

Finally, take a super-horizontal flag of holomorphic vector subbundles (16) and the corresponding S^1-invariant extended solution W_A. Take a meromorphic frame $s_1, \ldots, s_{d_{k-1}}$ of A_{k-1} such that, for each $i \in \{-r, \ldots, k-1\}$, $s_1, \ldots, s_{d_i}$ is a meromorphic frame of A_i and $s_1, \ldots, s_{d_i}, s_{d_i+1}, \ldots, s_{d_i+g_i}$ is a meromorphic frame of $A_i^{(1)}$. The extended solution W, with values in $U_\xi(\mathrm{U}(n))$ and $u_\xi(W) = W_A$, have Frenet frames of the form

$$\begin{aligned} X &= \operatorname{Span}\{s_1\lambda^{-r} + w_1\lambda^{-r+1}, \ldots, s_{d_{-r}}\lambda^{-r} + w_{d_{-r}}\lambda^{-r+1}\} \\ &\quad + \sum_{i=-r}^{k-2} \operatorname{Span}\{s_{d_i+g_i+1}\lambda^{i+1} + w_{d_i+g_i+1}\lambda^{i+2}, \ldots, s_{d_{i+1}}\lambda^{i+1} + w_{d_{i+1}}\lambda^{i+2}\}; \end{aligned} \tag{17}$$

where, for each $j \in \{1, \ldots, d_{k-1}\}$, w_j is a meromorphic section of $M \times H_+^n / \lambda^{r+k} H_+^n$. However, in the general case, these meromorphic sections w_j can not be chosen arbitrarily. For example, if s_1 is a constant section, $w_1' \lambda^{-r+2}$ becomes a section of W. So we have to impose that $p_0(w_1')$, whith p_0 the projection defined by (4), has no orthogonal component onto $A_{-r+2}^\perp$. In sections 5.4 and 5.5 we shall discuss in detail some examples.

5. Extended solutions in $\Omega\mathrm{SU}(n)$

5.1. **Grassmannian model for $\Omega\mathrm{SU}(n)$.** Consider the exterior product $\wedge$ of vectors in $\mathbb{C}^n$ and extend it to H^n as follows: if $f, g \in H^n$, then $(f \wedge g)(\lambda) = f(\lambda) \wedge g(\lambda)$. The loop group $\Omega\mathrm{U}(n)$ acts on $\wedge^n H^n$ in the natural way:

$$\gamma(f_1 \wedge \ldots \wedge f_n) := \gamma f_1 \wedge \ldots \wedge \gamma f_n = \det(\gamma)(f_1 \wedge \ldots \wedge f_n).$$

The Grassmannian model of $\Omega\mathrm{SU}(n)$ is given by:

Proposition 5.1. *A subspace $W \in \mathrm{Gr}^n$ corresponds to a loop in $\mathrm{SU}(n)$ if, and only if, it belongs to*

$$\mathrm{Gr}(\mathrm{SU}(n)) = \{W \in \mathrm{Gr}^n \,|\, \wedge^n W = \wedge^n H_+^n\}.$$

Proof. If $\gamma \in \Omega\mathrm{SU}(n)$, then it is clear that $\wedge^n W = \wedge^n H^n_+$, since $\Omega\mathrm{SU}(n)$ acts trivially on the $\wedge^n H^n$. Conversely, suppose that $\wedge^n W = \wedge^n H^n_+$. For each $\lambda \in \mathrm{S}^1$, consider the isomorphism given by the evaluation map at λ, $\mathrm{ev}_\lambda : W \ominus \lambda W \to \mathbb{C}^n$. Set $\gamma(\lambda) = \mathrm{ev}_\lambda \circ \mathrm{ev}_1^{-1}$, which is a loop of $\Omega\mathrm{U}(n)$ and, by Remark 2.1, verifies $W = \gamma H^n_+$. By hypothesis $\wedge^n(W \ominus \lambda W) \subset \wedge^n H^n_+$. Hence, $\mathrm{ev}_1^{-1}(e_1) \wedge \ldots \wedge \mathrm{ev}_1^{-1}(e_n) \in \wedge^n H^n_+$. Since

$$\begin{aligned}\det(\gamma)(e_1 \wedge \ldots \wedge e_n) &= \gamma(e_1) \wedge \gamma(e_2) \wedge \ldots \wedge \gamma(e_n) = \mathrm{ev}_\lambda \circ \mathrm{ev}_1^{-1}(e_1) \wedge \ldots \wedge \mathrm{ev}_\lambda \circ \mathrm{ev}_1^{-1}(e_n)\\ &= \mathrm{ev}_\lambda(\mathrm{ev}_1^{-1}(e_1) \wedge \ldots \wedge \mathrm{ev}_1^{-1}(e_n)),\end{aligned}$$

it follows that $\det(\gamma)$ is in H^1_+.

Now, since $\wedge^n \gamma H^n_+ = \wedge^n H^n_+$, we also have $\wedge^n H^n_+ = \wedge^n \gamma^{-1} H^n_+$. Hence, by the same argument as above, $\det(\gamma)^{-1}$ is in H^1_+. On the other hand, the fact that γ takes values in $\mathrm{U}(n)$ implies that $\det(\gamma)^{-1} = \overline{\det(\gamma)}$, which means that $\det(\gamma)^{-1}$ is also in H^1_-. This is possible if and only if $\det(\gamma)$ is constant. Since $\gamma(1) = e$, we must have $\det(\gamma) = 1$. $\square$

Proposition 5.2. If $\xi \in \mathfrak{I}(\mathrm{SU}(n)) \subset \mathfrak{I}(\mathrm{U}(n))$, then $U_\xi(\mathrm{SU}(n)) = U_\xi(\mathrm{U}(n))$.

Proof. Let $\gamma \in U_\xi(\mathrm{U}(n))$. Then $\gamma H^n_+ = \Psi \gamma_\xi H^n_+$ for some $\Psi \in \Lambda^+_{\mathrm{alg}}\mathrm{U}(n)$. Hence

$$\wedge^n \gamma H^n_+ = \wedge^n \Psi \gamma_\xi H^n_+ = \wedge^n \Psi W_\xi = \det(\Psi) \wedge^n W_\xi = \det(\Psi) \wedge^n H^n_+. \tag{18}$$

Since $\Psi \in \Lambda^+_{\mathrm{alg}}\mathrm{U}(n)$, $\det(\Psi)$ is polynomial in λ, hence $\wedge^n \gamma H^n_+ \subseteq \wedge^n H^n_+$. Conversely, from (18) we see that

$$\wedge^n H^n_+ = \det(\Psi^{-1}) \wedge^n \gamma H^n_+;$$

and since $\det(\Psi^{-1})$ is also polynomial in λ, we conclude that $\wedge^n H^n_+ \subseteq \wedge^n \gamma H^n_+$. $\square$

In particular, if $\xi \in \mathfrak{I}(\mathrm{SU}(n))$, all the extended solutions $W : M \setminus D \to U_\xi(\mathrm{SU}(n))$ arise from a Frenet frame of the form (17) without any further restriction on the choice of the meromorphic data.

5.2. **Canonical elements of** $\mathrm{SU}(n)$**.** Let E_i be the $(n \times n)$-matrix whose (i, i) entry is $\sqrt{-1}$ and whose other entries are all 0. The algebra of diagonal matrices

$$\mathfrak{t}^{\mathbb{C}} = \big\{ \sum a_i E_i : a_i \in \mathbb{C}, \sum a_i = 0 \big\}$$

is a Cartan subalgebra of $\mathfrak{su}(n)^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$. Let L_i in the dual of $\mathfrak{t}^{\mathbb{C}}$ be defined by $L_i(\sum a_j E_j) = \sqrt{-1} a_i$. The corresponding set of roots $\Delta \in \sqrt{-1}\mathfrak{t}^*$ is given by $\Delta = \{L_i - L_j : i, j = 1, \ldots, n\}$ and $\Delta^+ = \{L_i - L_j : i < j\}$ is a set of positive roots. The positive simple roots are then the roots of the form $\alpha_i = L_i - L_{i+1}$, with $i = 1, \ldots, n-1$, and the dual basis of $\mathfrak{t}$ is formed by the matrices

$$H_i = \frac{n-i}{n} E_1 + \ldots + \frac{n-i}{n} E_i - \frac{i}{n} E_{i+1} - \ldots - \frac{i}{n} E_n.$$

The Lie group $\mathrm{SU}(n)$ is precisely the simply connected Lie group with Lie algebra $\mathfrak{su}(n)$ and its center is $\mathbb{Z}_n$. The integer lattice $\mathfrak{I}(\mathrm{SU}(n)/\mathbb{Z}_n)$ is simply $\{\sum n_i H_i : n_i \in \mathbb{Z}\}$ and its I-canonical elements with respect to Δ^+ are the sums $\sum_{i \in I} H_i$, with $I \subseteq \{1, \ldots, n-1\}$. The I-canonical elements of $\mathrm{SU}(n)$ are not so easy to identify. We need to find the integral combinations of the elements H_i which are in $\mathfrak{I}'(\mathrm{G}) \cap \mathfrak{C}_I$ (that is, elements which are simultaneously integral combinations of the elements H_i and of the elements E_i) and are maximal with respect to $\preceq$. For example, when n is odd, it is easy to check that $\xi = H_1 +$

$H_2 + \ldots + H_{n-1}$ is the unique $[n-1]$-canonical element of $\mathrm{SU}(n)$ with respect to Δ^+, where $[p] = \{1, \ldots, p\}$. But when $n > 2$ is even there always exist more than one $[n-1]$-canonical element. The following lemma is useful in order to describe all canonical elements of $\mathrm{SU}(n)$:

Lemma 5.3. *The integer lattice $\mathfrak{I}(\mathrm{SU}(n))$ is invariant with respect to the linear isomorphism $\chi_1 : \mathfrak{t}^{\mathbb{C}} \to \mathfrak{t}^{\mathbb{C}}$ defined by $\chi_1(H_i) = H_{n-i}$, with $i \in [n-1]$. When $n = 2m+1$ is odd, $\mathfrak{I}(\mathrm{SU}(n))$ is also invariant with respect to the linear isomorphism $\chi_2 : \mathfrak{t}^{\mathbb{C}} \to \mathfrak{t}^{\mathbb{C}}$ defined by $\chi_2(H_i) = H_{2i}$ and $\chi_2(H_{n-i}) = H_{n-2i}$ if $i \in \{1, \ldots, m\}$.*

Proof. As we have observed before, an element of $\mathfrak{t}$ is in $\mathfrak{I}(\mathrm{SU}(n))$ if and only if its coefficients in E_i are integers. Hence, taking account that $H_i = E_1 + \ldots + E_i - \frac{i}{n}(E_1 + \ldots + E_n)$, an integer linear combination $\sum_{i=1}^{n-1} n_i H_i$ is in $\mathfrak{I}(\mathrm{SU}(n))$ if and only if $\sum_{i=1}^{n-1} \frac{in_i}{n}$ is an integer number, and this happens if and only if $\sum_{i=1}^{n-1} \frac{(n-i)n_i}{n}$ is integer. Hence $\mathfrak{I}(\mathrm{SU}(n))$ is invariant with respect to χ_1.

Assume now that $n = 2m+1$. In this case,

$$\sum_{i=1}^{n-1} \frac{in_i}{n} = \sum_{i=1}^{m} \frac{in_i}{n} + \sum_{i=1}^{m} \frac{(n-i)n_{n-i}}{n} = \sum_{i=1}^{m} \frac{in_i}{n} - \sum_{i=1}^{m} \frac{in_{n-i}}{n} + \sum_{i=1}^{m} n_{n-i}. \tag{19}$$

On the other hand, if we set

$$\sum_{i=1}^{n-1} n'_i H_i = \chi_2\Big(\sum_{i=1}^{n-1} n_i H_i\Big) = \sum_{i=1}^{m} n_i H_{2i} + \sum_{i=1}^{m} n_{n-i} H_{n-2i},$$

we get

$$\sum_{i=1}^{n-1} \frac{in'_i}{n} = \sum_{i=1}^{m} \frac{2in_i}{n} + \sum_{i=1}^{m} \frac{(n-2i)n_{n-i}}{n} = 2\sum_{i=1}^{m} \frac{in_i}{n} - 2\sum_{i=1}^{m} \frac{in_{n-i}}{n} + \sum_{i=1}^{m} n_{n-i}. \tag{20}$$

Comparing (19) with (20) we conclude that $\sum_{i=1}^{n-1} \frac{in'_i}{n} \in \mathbb{Z}$ if $\sum_{i=1}^{n-1} \frac{in_i}{n} \in \mathbb{Z}$, that is $\mathfrak{I}(\mathrm{SU}(n))$ is invariant with respect to χ_2. □

For each $i \in \{1, \ldots, n-1\}$, let m_i be the least positive integer which makes $m_i H_i$ and integral combination of the elements E_i. Since $H_i = E_1 + \ldots + E_i - \frac{i}{n}(E_1 + \ldots + E_n)$, m_i is precisely the denominator of the irreducible fraction equivalent to $\frac{i}{n}$. The canonical elements should then be sought among the elements of the finite set formed by the integral combinations $\sum_{i=1}^{n-1} n_i H_i$, with $n_i \in \{0, \ldots, m_i\}$, which are simultaneously integral combinations of the elements E_i. For general n and $I \subseteq \{1, \ldots, n-1\}$ it is too hard to list all the I-canonical elements. Next we will describe in detail the situation for the lower dimensional cases. We shall denote by π_i the orthogonal projection of $\mathbb{C}^n$ onto the one-dimensional vector subspace of $\mathbb{C}^n$ generated by the vector e_i.

5.3. **The case** $\mathrm{SU}(2)$**.** In this case there is a unique simple root α_1 with dual $H_1 = \frac{1}{2}(E_1 - E_2)$, which does not belong to the integer lattice $\mathfrak{I}(\mathrm{SU}(2))$. Consequently $\xi = 2H_1$ is the unique non-trivial canonical element – the corresponding homomorphism is $\gamma_\xi(\lambda) = \lambda^{-1}\pi_1 + \lambda\pi_1^\perp$. If $W : M \setminus D \to U_\xi(\mathrm{SU}(n))$ is a complex extended solution, then the corresponding S^1-invariant solution is given by $u_\xi(W) = \lambda^{-1}A + A + \lambda H_+^2$, where A is a holomorphic subbundle of $\underline{\mathbb{C}}^2$. It follows from the super-horizontality property that A is a constant bundle. Hence, we have $W = L + A + \lambda H_+^2$, where L is a holomorphic line bundle of $A\lambda^{-1} + A^\perp$, with $p_{-1}(L) \neq 0$ off

a discrete set of points, where p_{-1} is defined as in (4). That is, any harmonic map of finite uniton number from M into $\mathrm{SU}(2) \simeq \mathrm{S}^3$ arises from a constant direction A and a holomorphic line bundle of $\lambda^{-1}A + A^\perp \simeq \underline{\mathbb{C}}^2$. This agrees with the well known result by Calabi [4] that asserts that any locally minimal immersion of a surface in an odd dimensional sphere S^{2m-1} is contained in a hyperplane of $\mathbb{R}^{2m}$. In particular, no harmonic map of finite uniton number from M into $\mathrm{SU}(2) \simeq \mathrm{S}^3$ is full. This means that any such harmonic map takes values in a unit two-dimensional sphere $\mathrm{S}^2 \simeq \mathbb{C}\mathrm{P}^1$, that is, it corresponds to an holomorphic line bundle of $\underline{\mathbb{C}}^2$.

5.4. **The case** $\mathrm{SU}(3)$**.** We have two simple roots, α_1 and α_2, and three non-trivial canonical elements:

$$\xi_1 = H_1 + H_2 = E_1 - E_3; \quad \xi_2 = 3H_1 = 2E_1 - E_2 - E_3; \quad \xi_3 = 3H_2 = E_1 + E_2 - 2E_3.$$

The corresponding homomorphisms are given by

$$\gamma_{\xi_1}(\lambda) = \lambda^{-1}\pi_3 + \pi_2 + \lambda\pi_1; \;\; \gamma_{\xi_2}(\lambda) = \lambda^{-1}\left(\pi_2 + \pi_3\right) + \lambda^2\pi_1; \;\; \gamma_{\xi_3}(\lambda) = \lambda^{-2}\pi_3 + \lambda(\pi_2 + \pi_1).$$

If $W_{\xi_1} : M \setminus D \to U_{\xi_1}(\mathrm{SU}(3))$ is a complex extended solution, then the corresponding S^1-invariant solution is given by $u_{\xi_1}(W_{\xi_1}) = \lambda^{-1}B_3 + (B_2 \oplus B_3) + \lambda H^3_+$, where B_3 is a holomorphic line subbundle of the holomorphic vector bundle $B_2 \oplus B_3$ of rank 2. In order to construct all such extended solutions, and taking account the results of section 4.5, we start with a meromorphic section s_3 of $\underline{\mathbb{C}}^3$ and set $B_3 = \mathrm{Span}\{s_3\}$. If B_3 is not constant, we define $B_2 = B_3^{(1)} \ominus B_3$, take an arbitrary holomorphic section w_3 of $\underline{\mathbb{C}}^3$ and set $X_{\xi_1} = \mathrm{Span}\{\lambda^{-1}s_3 + w_3\}$. If B_3 is constant, we take an arbitrary meromorphic section s_2. By adding a constant if necessary, s_2 and s_3 are linearly independent and we set $B_2 = \mathrm{Span}\{s_2, s_3\} \ominus B_3$. Take an arbitrary holomorphic section w_3 of $\underline{\mathbb{C}}^3$ and set $X_{\xi_1} = \mathrm{Span}\{\lambda^{-1}s_3 + w_3, s_2\}$. In both cases, X_{ξ_1} is a Frenet frame for an extended solution $W_{\xi_1} : M \setminus D \to U_{\xi_1}(\mathrm{SU}(3))$.

If $W_{\xi_2} : M \setminus D \to U_{\xi_2}(\mathrm{SU}(3))$ is a complex extended solution, then the corresponding S^1-invariant solution is given by

$$u_{\xi_2}(W_{\xi_2}) = \lambda^{-1}(B_2 \oplus B_3) + (B_2 \oplus B_3) + \lambda(B_2 \oplus B_3) + \lambda^2 H^3_+.$$

By the super-horizontality property, $B_2 \oplus B_3$ is constant, and consequently B_1, the orthogonal complement of $A_2 \oplus A_3$, is also constant. In order to construct all such extended solutions, fix a two-dimensional subspace with basis elements s_2 and s_3. Take arbitrary meromorphic sections w_2 and w_3 of $\underline{\mathbb{C}}^3 + \lambda\underline{\mathbb{C}}^3$ with $\pi_{B_1}(p_0(w_2))$ and $\pi_{B_1}(p_0(w_3))$ constants, where p_0 is the projection defined by (4). Then $X_{\xi_2} = \mathrm{Span}\{s_2\lambda^{-1} + w_2, s_3\lambda^{-1} + w_3\}$ is a Frenet frame for an extended solution $W_{\xi_2} : M \setminus D \to U_{\xi_2}(\mathrm{SU}(3))$, and all such extended solutions arise in this way. We observe that, taking account Lemma 3.17 and Proposition 3.18 of [13], the corresponding harmonic map has *uniton number* one, in the sense that it admits an extended solution with values in $\Omega\mathrm{U}(3)$ of the form $\pi_V + \lambda\pi_V^\perp$, with V a holomorphic subbundle of $\underline{\mathbb{C}}^3$. The case W_{ξ_3} is similar.

5.5. **The cases** $\mathrm{SU}(4)$ **and** $\mathrm{SU}(5)$**.** Table 1 shows all the non-trivial canonical elements of $\mathrm{SU}(4)$ and $\mathrm{SU}(5)$ up to the symmetries χ_1, χ_2 of Lemma 5.3.

We describe how to construct, for $\xi_1 = H_1 + 2H_2 + H_3 = 2E_1 + E_2 - E_3 - 2E_4$, all the extended solutions $W_{\xi_1} : M \setminus D \to U_{\xi_1}(\mathrm{SU}(4))$. We have $\gamma_{\xi_1} = \lambda^{-2}\pi_4 + \lambda^{-1}\pi_3 + \lambda\pi_2 + \lambda^2\pi_1$, and, consequently,

$$u_{\xi_1}(W_{\xi_1}) = \lambda^{-2}B^1_4 + \lambda^{-1}\left(B^1_3 \oplus B^1_4\right) + \left(B^1_3 \oplus B^1_4\right) + \lambda\left(B^1_2 \oplus B^1_3 \oplus B^1_4\right) + \lambda^2 H^4_+,$$

$\mathrm{SU}(n)$	$\lvert I\rvert = n-1$	$\lvert I\rvert = n-2$	$\lvert I\rvert = n-3$	$\lvert I\rvert = n-4$
$n=4$	$H_1+2H_2+H_3$	$2H_1+H_2$	$4H_1$	
	$3H_1+H_2+H_3$	H_1+H_3	$2H_2$	
$n=5$	$H_1+H_2+H_3+H_4$	$H_1+H_2+4H_3$	H_1+2H_2	$5H_1$
		$H_1+3H_2+H_3$	$3H_1+H_2$	
		$2H_1+H_2+2H_3$	H_1+H_4	
		$3H_1+2H_2+H_3$		
		$5H_1+H_2+H_3$		

TABLE 1. Canonical elements for SU(4) and SU(5).

where each vector subbundle B_i^1 has rank one. The harmonic map associated to this S^1-invariant extended solution is given by

$$\varphi_1 = \pi_{B_1^1\oplus B_4^1} - \pi_{B_2^1\oplus B_3^1}.$$

By super-horizontality, $B_3^1\oplus B_4^1$ is a constant bundle. So, in order to construct all such extended solutions, we start by fixing a two-dimensional vector subspace V of $\mathbb{C}^4$ generated by constant vectors u and v. Next take meromorphic sections:

(1) s_1 of $\underline{V}$, and set $B_4^1 = \mathrm{Span}\{s_1\}$ and $B_3^1 = \underline{V}\ominus B_4^1$;
(2) s_3 of $\underline{V}^\perp$, and set $B_2^1 = \mathrm{Span}\{s_3\}$ and $B_1^1 = \underline{V}^\perp\ominus B_2^1$;
(3) w_1 of $M\times H_+^4/\lambda^3 H_+^4$.

If B_4^1 is not constant, we can write $s_1'' = g_1 s_1 + g_2 s_1'$ for some meromorphic functions g_1 and g_2 on M, with $g_1'g_2 - g_2'g_1 \neq 0$. In this case, $X = \mathrm{Span}\{\lambda^{-2}s_1 + \lambda^{-1}w_1, \lambda s_3\}$ is a Frenet frame for an extended solution with values in $U_{\xi_1}(\mathrm{SU}(4))$ if and only if

$$\pi_{B_1^1}\circ p_0(w_1'' - g_1 w_1 - g_2 w_1') = 0.$$

For $\xi_2 = 3H_1+H_2+H_3 = 3E_1 - E_3 - 2E_4$, we have $\gamma_{\xi_2} = \lambda^{-2}\pi_4 + \lambda^{-1}\pi_3 + \pi_2 + \lambda^3\pi_1$, and, consequently,

$$u_{\xi_2}(W_{\xi_2}) = \lambda^{-2}B_4^2 + \lambda^{-1}\left(B_3^2\oplus B_4^2\right) + \left(B_2^2\oplus B_3^2\oplus B_4^2\right) + \lambda\left(B_2^2\oplus B_3^2\oplus B_4^2\right) + \lambda^2\left(B_2^2\oplus B_3^2\oplus B_4^2\right) + \lambda^3 H_+^4.$$

The harmonic map associated to this S^1-invariant extended solution is given by

$$\varphi_2 = \pi_{B_2^2\oplus B_4^2} - \pi_{B_1^2\oplus B_3^2}.$$

Although φ_1 and φ_2 are both of the form $\pi_E - \pi_E^\perp$, with E a rank two vector subbundle of $\underline{\mathbb{C}}^4$, these vector bundles exhibit distinct geometrical behaviours. For example, whereas $E = B_2^2\oplus B_4^2$ has always constant (2)-osculating bundle, both $E = B_1^1\oplus B_4^1$ and $E^\perp = B_2^1\oplus B_3^1$ can have non-constant (2)-osculating bundle.

5.6. **Symmetric canonical elements of** $\mathrm{SU}(n)$**.** All the compact inner symmetric spaces of $\mathrm{SU}(n)$ are complex grassmannians $\mathrm{Gr}(k,n)$. The embedding ι of $\mathrm{Gr}(k,n)$ as a connected component of $\sqrt{e}$ is given by $\iota(V) = \pi_V - \pi_V^\perp$, where $V\in\mathrm{Gr}(k,n)$ is k-dimensional subspace of $\mathbb{C}^n$. There exists no non-trivial symmetric canonical element for SU(2). Table 2 presents all non-trivial symmetric canonical elements for $\mathrm{SU}(n)$, with $n = 3,4,5$, up to the symmetries χ_1,χ_2. As before, for each $i\in\{1,\dots,n-1\}$, let m_i be the least positive integer which

makes m_iH_i and integral combination of the elements E_i. The symmetric canonical elements should then be sought among the elements of the finite set formed by the integral combinations $\sum_{i=1}^{n-1} n_iH_i$, with $n_i \in \{0,\dots,2m_i-1\}$, which are simultaneously integral combinations of the elements E_i.

We use the usual hermitian inner product on $\mathbb{C}^n$ to identify $\mathrm{Gr}(k,n)$ with $\mathrm{Gr}(n-k,n)$. It is easy to check that, for $\xi \in \mathfrak{I}(\mathrm{SU}(n))$, $N_\xi = N_{\chi_1(\xi)}$. However, in general, the symmetric space $N_{\chi_2(\xi)}$ does not coincide with N_ξ. For example, in SU(5) the two following situations can occur: for $\xi = 5H_1$, we have $\chi_2(\xi) = 5H_2$, $N_\xi = \mathrm{Gr}(1,5)$ and $N_{\chi_2(\xi)} = \mathrm{Gr}(2,5)$; on the other hand, for $\eta = 3H_1 + H_2 + 5H_3$, we have $N_\eta = N_{\chi_2(\eta)} = \mathrm{Gr}(2,5)$.

$\mathrm{Gr}(k,n)$	$\lvert I\rvert = n-1$	$\lvert I\rvert = n-2$	$\lvert I\rvert = n-3$	$\lvert I\rvert = n-4$
$k=1, n=3$	H_1+H_2 $4H_1+H_2$	$3H_1$		
$k=2, n=4$	$3H_1+H_2+H_3$	$2H_1+H_2$ H_1+H_3		
$k=1, n=5$	$4H_1+2H_2+H_3+H_4$	$H_1+H_2+4H_3$	H_1+2H_2 H_1+7H_2 $3H_1+H_2$ H_1+6H_4	$5H_1$
$k=2, n=5$	$H_1+H_2+H_3+H_4$ $2H_1+3H_2+H_3+H_4$ $H_1+H_2+H_3+6H_4$	$H_1+H_2+9H_3$ $H_1+3H_2+H_3$ $H_1+8H_2+H_3$ $2H_1+H_2+2H_3$ $3H_1+H_2+5H_3$ $3H_1+2H_2+H_3$ $5H_1+H_2+H_3$	$4H_1+3H_2$ $8H_1+H_2$ H_1+H_4	

TABLE 2. Symmetric canonical elements for $\mathrm{SU}(n)$, with $n \leq 5$.

We describe how to construct, for $n=4$, $k=2$ and $\xi_1 = 2H_1 + H_2 = 2E_1 - E_3 - E_4$, all the extended solutions $W_{\xi_1} : M \setminus D \to U^{\mathcal{I}}_{\xi_1}(\mathrm{SU}(4))$. We have $\gamma_{\xi_1} = \lambda^{-1}(\pi_3+\pi_4) + \pi_2 + \lambda^2\pi_1$ and, consequently,

$$u_{\xi_1}(W_{\xi_1}) = \lambda^{-2}(B^1_4 \oplus B^1_3) + (B^1_4 \oplus B^1_3 \oplus B^1_2) + \lambda(B^1_4 \oplus B^1_3 \oplus B^1_2) + \lambda^2 H^4_+,$$

where each vector subbundle B^1_i has rank one. The harmonic map associated to this S^1-invariant extended solution is given by $\varphi_1 = \pi_{B^1_1\oplus B^1_2} - \pi_{B^1_3\oplus B^1_4}$. So, take meromorphic sections s_1, s_2, w_1, w_2 of $\underline{\mathbb{C}}^4$ and set $B_3 \oplus B_4 = \mathrm{Span}\{s_1, s_2\}$. Assuming that this vector bundle is not constant, $X_{\xi_1} = \mathrm{Span}\{\lambda^{-1}s_1 + \lambda w_1, \lambda^{-1}s_2 + \lambda w_2\}$ will be a Frenet frame for an extended solution with values in $U^{\mathcal{I}}_{\xi_1}(\mathrm{SU}(4))$. Moreover, all such extended solutions, with $B_3 \oplus B_4$ not constant, arise in this way.

Consider also the case $n=4$, $k=2$ and $\xi_2 = H_1 + H_3 = E_1 - E_4$. We have $\gamma_{\xi_2} = \lambda^{-1}\pi_4 + (\pi_3 \oplus \pi_2) + \lambda\pi_1$ and, consequently, $u_{\xi_2}(W_{\xi_2}) = \lambda^{-1}B^2_4 + B^2_4 \oplus B^2_3 \oplus B^2_2 + \lambda H^4_+$. The harmonic map associated to this S^1-invariant extended solution is given by $\varphi_2 = \pi_{B^2_2\oplus B^2_3} - \pi_{B^2_1\oplus B^2_4}$. Observe that, in this case, we only have S^1-invariant extended solutions since $U^{\mathcal{I}}_{\xi_2}(\mathrm{SU}(4)) = \Omega_{\xi_2}$.

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